

Numerical Method for Solution of the Integral Equation of the First Kind — Application to Analysis of Plasma Density Profile

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A numerical method based on nonlinear programming is applied to the solution of an integral equation of the first kind. Instead of determining the values of the dependent variable at the prescribed values of the independent variable, this method solves the equation by determining values of the independent variable for prescribed values of the dependent variable, with some appropriate constraining conditions. By this method, unphysical oscillations, likely to appear in the solution of the integral equation of the first kind, are eliminated, and various integral equations of the first kind, appearing in the analyses of experimental data, can be solved with comparative ease. To demonstrate the usefulness of this method, a data set of line densities of a plasma column, integrated along paths across a region of interest, is inversely transformed, and a density profile of the plasma column is successfully obtained.

1. INTRODUCTION

In the fields of experimental plasma physics or nuclear fusion research, it is often necessary to solve an integral equation of the first kind such as,

$$\phi(x) = \int_a^b z(y) K(x, y) dy. \quad (1)$$

A typical example of the problem is solution of the Abel's integral equation¹ which is needed to obtain profiles of radiation sources or electron densities in a cylindrical plasma column from measured integrated quantities along lines across the region of interest. Numerical methods to solve this kind of linear integral equation have been intensively investigated by many authors [1] and it is well known that in solving realistic problems; i.e., in analyzing a set of experimental data, at least two difficulties are usually encountered. One is the mathematical problem associated

¹ Though the Abel's integral equation is of Volterra type, an equation of this type can be transformed to that of Fredholm type by modifying the kernel of the equation. Therefore, we are mainly concerned about the equation of Fredholm type in the remainder of this paper.

with the properties of the integral equation of the first kind [2], which is due to the fact that the matrix generated from the integral equation is ill-conditioned. Therefore, slight errors in the kernel $K(x, y)$ or in the given function $\phi(x)$ cause extremely large errors in the solution $z(y)$, and thus the equation cannot be uniquely determined. The other difficulty concerns mathematical modeling of the physical problem. Experimentally we often find a case where a function defined in a two-dimensional space should be determined, e.g., when the system is not sufficiently symmetric to be reduced to a one-dimensional problem, such as in the analysis of a toroidal plasma column approximated by a cylindrical one. To determine an asymmetric density profile in the cross section of a cylindrical plasma column from a data set of line densities (microwave phase shifts) along various paths [3] for example, the integral equation to be solved is expressed as

$$\phi(\rho, \psi) = \int_0^R \{n(r, \psi + \cos^{-1}(\rho/r)) + n(r, \psi - \cos^{-1}(\rho/r))\} r dr / (r^2 - \rho^2)^{1/2}, \quad (2)$$

where ρ and ψ are the distance of the path from the center and the angle of the path from the y -axis, respectively, R is the radius of the column, ϕ is the phase shift data. As shown in the next section, under some appropriate approximation the above equation is reduced to the following nonlinear integral equation where the unknown function can be defined even in a one-dimensional space,

$$\phi(x) = \int_a^b K(x, y, z(y)) dy. \quad (3)$$

To overcome the first difficulty numerous numerical algorithms have been proposed and successfully applied to many kinds of problems. Most of these algorithms essentially are based on the solution of a set of simultaneous linear equations, utilizing the least squares method and some appropriate constraining conditions. Conversely, generally little has been done to solve the second difficulty [3] inasmuch as different types of nonlinearity are apt to appear in the course of constructing a practical numerical model, and therefore general treatment of the problem is often very difficult. Thus a new algorithm for solution of the nonlinear integral equation should be devised which would consider the physical properties of the problem for each new problem encountered. In the case of the above examples, nonlinearity appears when the independent and dependent variables are interchanged in order to take account of the distortions of the profile from the symmetric one in the Abel's integral equation and the quantities characterizing the distortion are chosen as unknown variables.

In the remainder of this article, we restrict ourselves to the problems of determining the density profile of a plasma column from a set of microwave phase shift data, where all contour lines of the profile are assumed to be circular, and a slight

asymmetry of the system is introduced by the displacements of these contour lines. In the usual approach to solution, contour lines of the profile are represented by concentric circles and the densities on these circles are essentially determined by solving the simultaneous linear equations which are derived by discretizing the original integral equation. By this method, both the shapes and positions of the contour lines of the profile are prescribed beforehand. Therefore, in the cases where the contour lines are displaced, the method fails to find a well-approximated density profile. To allow for displacement, some appropriate iteration procedures can also be incorporated [3], to take the displacement into account, but it is difficult to establish a general scheme for the iteration procedures and determine an appropriate initial guess values.

In this paper we describe a solution method which uses numerical codes of nonlinear programming algorithms by which one can minimize (or maximize) a function with an arbitrary nonlinear dependence on unknown parameters [4]. These codes are widely applicable in physics and engineering in the solution of nonlinear differential equations [5] or in designing a system where the objective is the determination of a nonlinear function of parameters [6]. In our case, the integral equation is discretized by taking the positions and radii of the contour circles as independent variables, instead of the values of the densities to be assigned to the contour circles. While this choice of parameters solves the above described second difficulty, it complicates the solution of the problem within the framework of the linear algebra, inasmuch as the original integral equation is reduced to a nonlinear function of these parameters. However, once the problem is transformed into an optimization problem, the numerical codes of nonlinear programming are very easily applied, since the mathematical properties of these codes, having been investigated in detail, can subsequently be utilized without each users extensive knowledge of them.

Section 2 describes the method of solution, while Section 3 examples of the results, obtained by applying the method to the analysis of the microwave data, are shown. Discussions on the method are presented in Section 4.

2. METHOD OF SOLUTION

The two kinds of difficulties described in the previous section are overcome by the following considerations. First, the least squares method is adopted to smooth a set of experimental data. Generally, this procedure is not sufficient to obtain a reasonably smooth solution of the integral equation of the first kind, and therefore, we introduce some constraining conditions concerning the shape of the density profile. In practical usage, these conditions are expressed in terms of the quantity

of extrema existing on the density profile. In the following discussions, we consider the case of the existence of only one extremum point. In other words, density decreases monotonically from the center of the profile to the periphery. Secondly, as previously mentioned, density of each contour line normalized with respect to the peak value is assigned beforehand, and the contour line of the profile is assumed to be circular. Thus, the position and radius of each contour circle and the peak density of the profile are chosen to be independent variables of the problem. Therefore, the number of unknown variables to approximate the density profile by N contour circles is $2N + 1$ (N radii, N displacements, and the peak density of the profile).

Denoting the density profile and phase shift of the microwave along a line x (=line density at x) by $n(x, y)$ and $\phi(x)$, respectively, in the Cartesian coordinate system (Fig. 1), the relation between $n(x, y)$ and $\phi(x)$ is expressed as

$$\phi(x) = \int_{-Y(x)}^{Y(x)} n(x, y) dy = 2 \int_0^{Y(x)} n(x, y) dy, \quad (4)$$

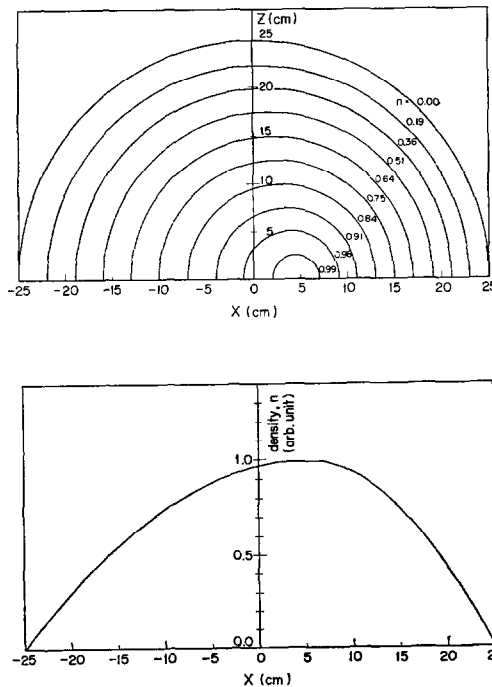


FIG. 1. Test data for the numerical code. Contour lines of the density profile (a) and the density on the median plane, $y = 0$ (b).

where $(x, \pm Y(x))$ are the intersections of the microwave path with the outermost contour circle. As only displacements in the x direction are detectable, the displacements along the y direction are meaningless, and a contour line for the density n is expressed as

$$(x - d(n))^2 + y^2 = r(n)^2, \quad (5)$$

where $d(n)$ and $r(n)$ are the displacement and radius of the contour circle of density n , respectively. Interchanging the independent variable between r and n in Eq. (4), one can easily obtain the equation,

$$\phi(x) = 2 \int_{n(x-d)}^0 \frac{r(n) - [d(n) - r(n)](d/dn) d(n)}{(r(n)^2 - [x - d(n)]^2)^{1/2}} \cdot n \frac{d}{dn} r(n) dn. \quad (6)$$

If the displacement $d(n)$ is identically zero, the contour lines are concentric circles, and Eq. (6) is reduced to a nonlinear integral equation of $r(n)$ as

$$\phi(x) = \int_{n(x)}^{n(R)} \frac{(d/dn) r(n)^2}{(r(n)^2 - x^2)^{1/2}} \cdot n dn, \quad (7)$$

which is equivalent to the usual representation of the Abel's integral equation,

$$\phi(x) = 2 \int_x^R \frac{n(r)}{(r^2 - x^2)^{1/2}} \cdot r dr. \quad (8)$$

The solution is roughly described as follows. The right-hand side of Eq. (6) is discretized and expressed by $2N + 1$ variables, the objective function of nonlinear programming is defined by the sum of squares of the differences between the left- and right-hand sides of Eq. (6), and then the objective function is minimized with respect to the above $2N + 1$ variables. We now describe the procedure in more detail. The radius, displacement, and density of the j th contour circle are denoted by r_j , d_j , and $n_j = n_0 \cdot S_j$, respectively, where n_0 is the density of the N th contour circle and, therefore, $S_N = 1$. To carry out the integration along a path between the j th and $(j + 1)$ th contour circles (the j th region), the density profile in this region is assumed to be well represented by an oblique frustum of cone approximated by a pile of K_j cylinders. The radius, displacement, and height (density) of the k th cylinder in this j th region are denoted by r_j^k , d_j^k , and $n_j^k = n_0 \cdot S_j^k$, respectively, and are given by

$$r_j^k = r_j + \frac{r_{j+1} - r_j}{K_j - 1} \cdot (k - 1), \quad (9)$$

$$d_j^k = d_j + \frac{d_{j+1} - d_j}{K_j - 1} \cdot (k - 1), \quad (10)$$

$$S_j^k = S_j + \frac{S_{j+1} - S_j}{K_j - 1} \cdot (k - 1), \quad (11)$$

where it is assumed that $d_j^1 = d_j$, $r_j^1 = r_j$, $S_j^1 = S_j$, $d_j^{K_j} = d_{j+1}$, $r_j^{K_j} = r_{j+1}$, and $S_j^{K_j} = S_{j+1}$. If the phase shifts of the microwave are measured at $x = x_m$ ($m = 1, 2, \dots, M$), the objective function of nonlinear programming $F = F(r_1, \dots, r_N, d_1, \dots, d_N, n_0)$ is expressed as

$$F = \sum_{m=1}^M w_m (\hat{\phi}_m - \phi_m)^2, \quad (12)$$

where w_m , $\hat{\phi}_m$, and ϕ_m are the weighting factor, the calculated and measured values of the phase shift at $x = x_m$, respectively. The calculated value of the phase shift $\hat{\phi}_m$ is derived as

$$\hat{\phi}_m = 2n_0 \sum_{j=1}^M \hat{\phi}_{mj}, \quad (13)$$

where

$$\hat{\phi}_{mj} = \sum_{k=1}^{K_j-1} S_j^k (y_{mj}^k - y_{mj}^{k+1}), \quad \text{for } r_j^2 \geq (x_m - d_j)^2, \quad (14)$$

$$= 0, \quad \text{for } r_j^2 < (x_m - d_j)^2, \quad (15)$$

where y_{mj}^k is the y coordinate of the intersection of the k th circle in the j th region with the line $x = x_m$ and it is expressed as

$$y_{mj}^k = ((r_j^k)^2 - (x_m - d_j^k)^2)^{1/2}, \quad \text{for } (r_j^k)^2 \geq (x_m - d_j^k)^2, \quad (16)$$

$$= 0, \quad \text{for } (r_j^k)^2 < (x_m - d_j^k)^2, \quad (17)$$

and

$$y_{mN}^{K_N} = 0. \quad (18)$$

Next, a set of constraining conditions is introduced to suppress unphysical oscillations which are apt to appear in the solution of the integral equation of the first kind. For example, a circle with a higher number should be always inside the circles with lower number, as expressed by the following inequalities,

$$\begin{aligned} -r_{\text{lim}} < d_1 - r_1 < \dots < d_N - r_N \\ < d_N + r_N < \dots < d_1 + r_1 < r_{\text{lim}}, \end{aligned} \quad (19)$$

where r_{lim} is the radius of the limiting circle. The limiting circle is introduced to enable all the contour circles to be confined within a physically permissible region, and the solution should not diverge during the course of iterations. However, if

sufficiently large number of data points are used, the limiting circle is not necessary and the radius of the outermost contour circle is uniquely determined as shown in the following section.

To minimize the objective function F , we utilized the numerical code developed by Nelder and Mead (the simplex method) [7] among many codes of nonlinear programming. This code does not require analytical expression for the derivatives of the objective function, which is very advantageous in such a case as the example in this paper where the derivatives of the objective function with respect to the unknown variables are not continuous function of them. Moreover the simplex method is very easily applicable in the case where many constraining conditions are incorporated with the minimization problem.

3. RESULTS OF CALCULATION

To examine the convergence and accuracy of the solution we carried out an inverse transformation of the data set of microwave phase shift which were prepared numerically from the following simple density profile,

$$r = r(S) = R(1 - S)^{1/2}, \quad (20)$$

$$d = d(S) = d_0 S^{1/2}, \quad (21)$$

$$n = n(S) = n_0 S, \quad (22)$$

where R , d_0 , and n_0 are the maximum values of the radius, displacement, and density of the contour circles, respectively, and S is the parameter ($0 \leq S \leq 1$) which identifies each contour circle and corresponds to the density normalized with respect to the maximum value. As is seen from Fig. 1, this density profile is produced from an axisymmetric parabolic density distribution by displacing the contour circles slightly in the direction of x -axis. The phase shift data for the microwave paths parallel to the y -axis are derived from Fig. 1 and are shown in Fig. 2.

First, using all the data points (\cdot) in Fig. 2 and setting the radius of the limiting circle to be 25 cm, the dependence of the solution on the number of the contour circles was investigated. Figure 3 shows solutions for $N = 3, 4$, and 5 with the exact solution (broken line). The solution for $N = 6$ is shown in Fig. 4 where the initial guess value is also added by a dotted line. The deviations of calculated phase shift from the measured one are presented for the cases of $N = 4$ and 6 in Fig. 5. This figure shows that the deviations of the calculated data are within the experimental error (several percent) even for such a small number of parameters as $N = 4$. The convergence of the calculation for $N = 6$ is presented in Fig. 6 as a

function of the number of iteration. Though all the solutions for $N = 3, 4,$ and 5 (Fig. 3) were obtained successfully by choosing a set of normalized density $\{S_i\}$ to be an arithmetic sequence, for larger N (in this case $N = 6$) this choice of $\{S_i\}$ failed to lower the value of the objective function sufficiently. In such situation better result was obtained by choosing $\{S_i\}$ so that the contour lines are as equally

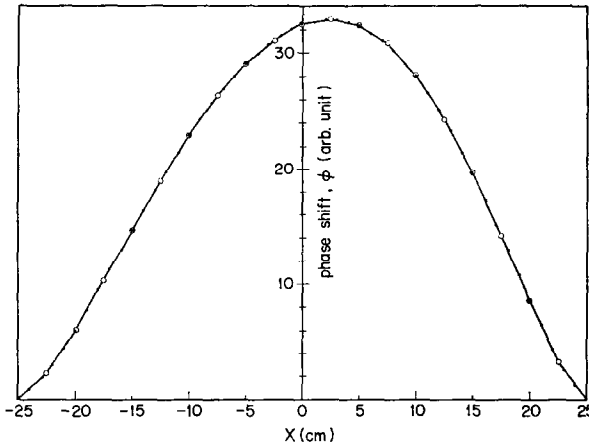


FIG. 2. The phase shift data corresponding to the density profile of Fig. 1.

spaced as possible. The result presented in Fig. 4 was obtained by choosing $\{S_i; i = 1, \dots, 6\}$ to be 0.0, 0.35, 0.60, 0.80, 0.93, and 1.0, respectively. Two cases for the different choice of $\{S_i\}$ are illustrated in Fig. 6 to demonstrate the effect of the choice of $\{S_i\}$ sequence on the convergence of the calculation.

Secondly, by removing the limiting circle and reducing the number of the data points we investigated whether the density profile of the peripheral region of the plasma column is reproduced correctly from the experimentally obtainable data set. The results (Fig. 7) show comparatively accurate shapes of the profile were obtained as long as about half of the original data points remain, when the observation points were removed only from one side of the plasma boundary. However, when the observation points were removed from both sides, half of the original data were not sufficient to reproduce the original profile correctly and the results show considerably distorted profiles.

Finally, in addition to the analyses of the above artificial density profile, an example of the experimental data obtained in the JFT-2 (JAERI tokamak) experiment [8] was analyzed by this method. In this case we approximated the density profile by eight contour circles, which means that the problem is reduced to a minimization of a nonlinear function with 17 parameters. While in JFT-2, only

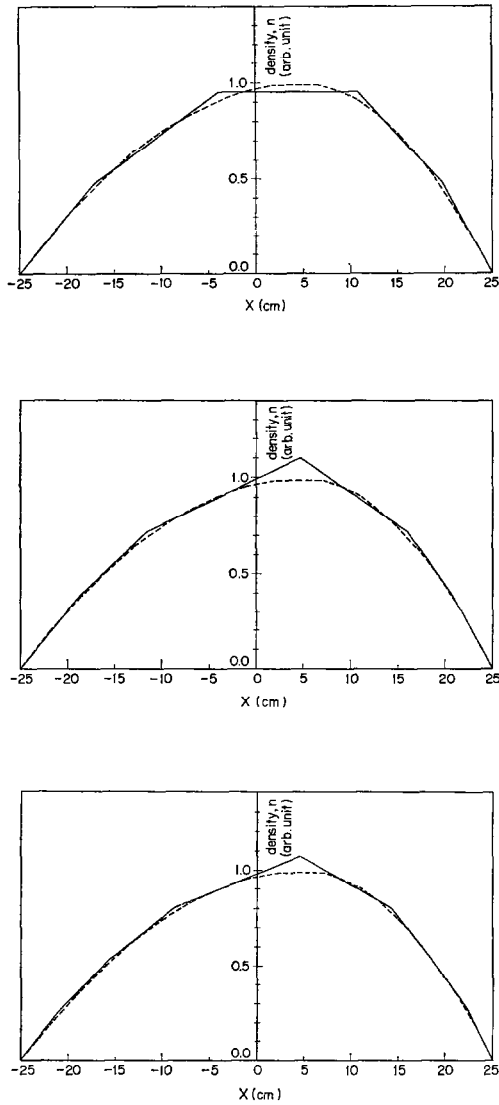


FIG. 3. The solutions of the integral equation (Eq. (6)) for the data of Fig. 2. The exact solution is also shown by a broken line. (a) $N = 3$, (b) $N = 4$, and (c) $N = 5$.

eight data points for vertical microwave paths are normally available in the region from $x = -18$ cm to $x = 24$ cm (Fig. 8, \circ), which are insufficient to determine 17 unknown parameters. Therefore we increased the data by linear interpolation (Fig. 8, solid line) and made 43 data points. Then the data set was inversely transformed to the density profile (Fig. 9). The phase shifts calculated from the profile of Fig. 9 are also shown by dots in Fig. 8. This figure shows that the calculated phase shifts agree well with the experimental ones within the experimental error.

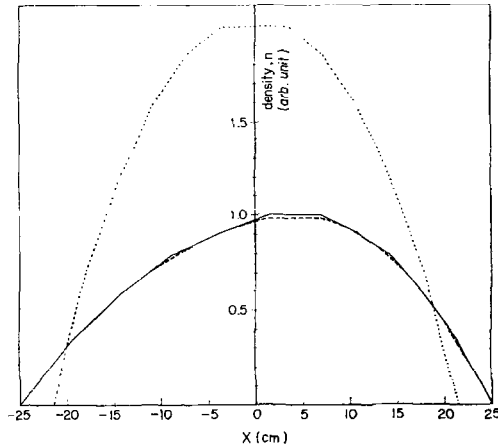


FIG. 4. The solution of the integral equation (Eq. (6)) for the data of Fig. 2, where $N = 6$. The initial guess value and the exact solution are shown by dotted and broken lines, respectively.

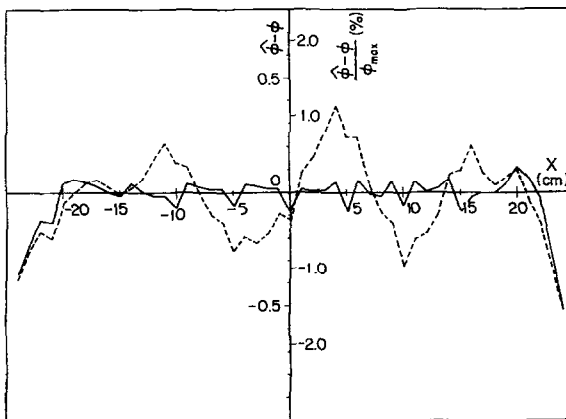


FIG. 5. The deviations of the calculated phase shift from the prepared one for $N = 4$ (broken line) and $N = 6$ (solid line).

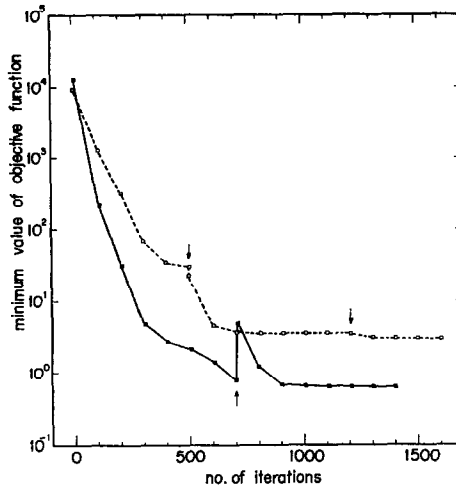


FIG. 6. The convergence of the calculation for $N = 6$. The broken and solid lines show the results for $\{S_i\} = \{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ and $\{S_i\} = \{0.0, 0.35, 0.60, 0.80, 0.93, 1.0\}$, respectively. Arrows indicate the points where the calculations are restarted by using rounded-off data of the preceding calculations.

4. DISCUSSION

In the previous section it is shown that a new numerical method based on non-linear programming is successfully applied to solution of an integral equation of the first kind. This numerical method can be applied to many kinds of problems which can be reduced to solution of the integral equation of this kind, though in this paper usefulness of the method is shown only for the limited class of application, i.e., the inversion of the data set obtained by the microwave interferometer with several restrictions. Two of these restrictions and the method to remove them are mentioned in the following. First, we assumed that all the contour lines were circular. This kind of restrictions which prescribe shapes of contour lines in a two-dimensional space was inevitable in this example because all the microwave paths were parallel to the y -axis and in order to determine the density profile some assumptions should be introduced concerning the profile in the y direction. If experimental data which represent integrated quantities along different directions are available, the assumption of circular contour line can be, of course, removed. Moreover, if the experimental data are obtained in a two-dimensional space, e.g., (ρ, ψ) space as in Eq. (2), arbitrary shaped profile will be reproduced in principle. Secondly, the analysis in this paper is restricted to the profiles which have only one peak and no valley. But the method is easily applied to the analysis of a

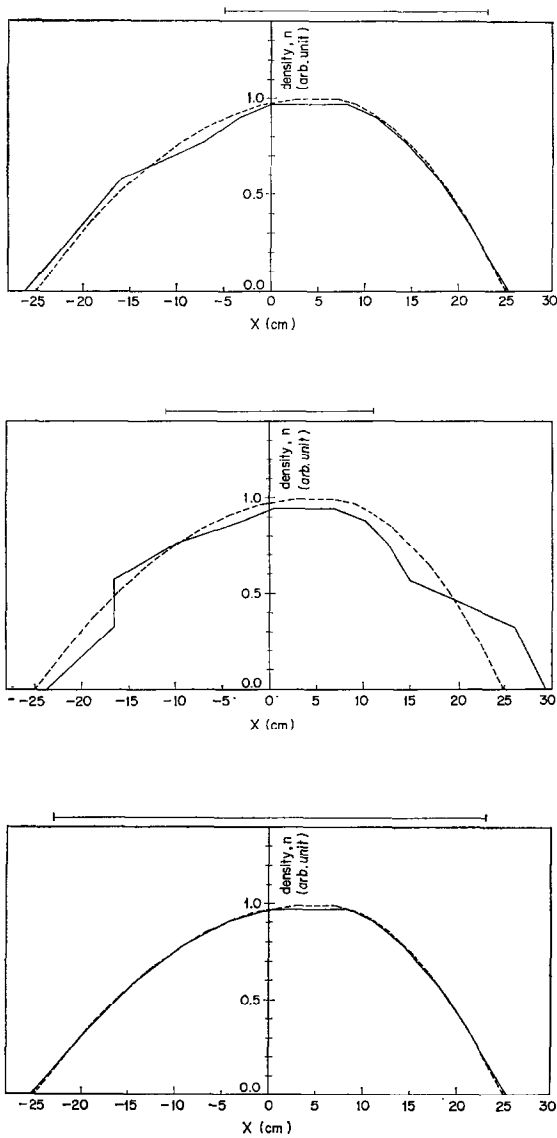


FIG. 7. The effects of the width of the observation region on the shape of the density profile obtained by the method. The width of the observation region is indicated above each subfigure by a solid line segment.

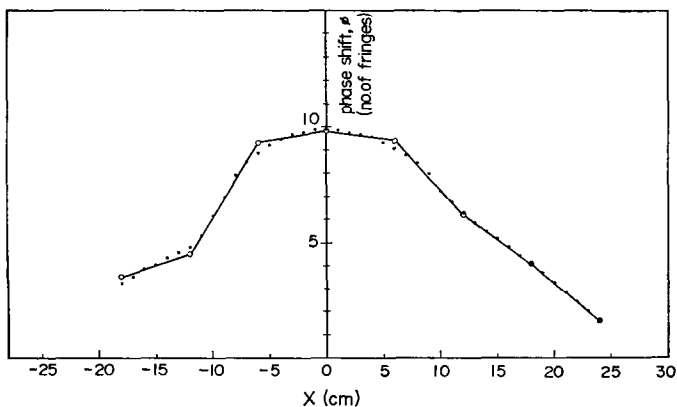


FIG. 8. An example of the phase shift data obtained in the JFT-2 experiment (○). The calculated phase shifts for the profile in the next figure are also shown by dots.

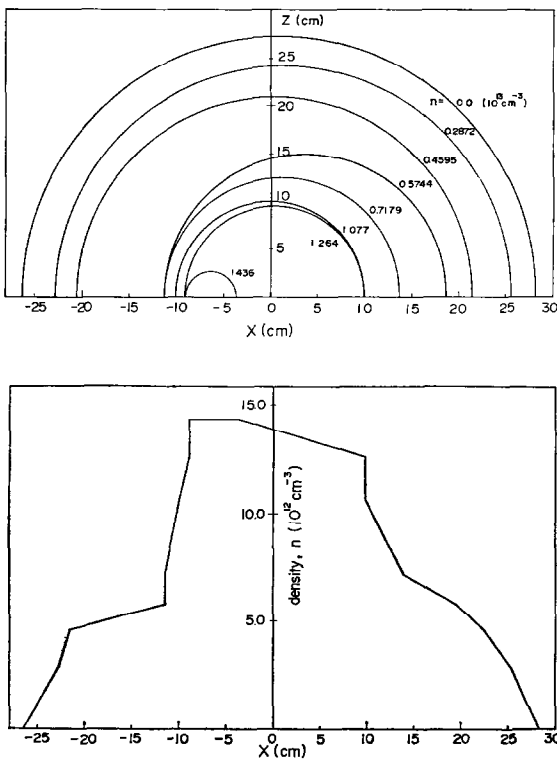


FIG. 9. Contour lines of the density profile (a) and the density on the median plane (b) calculated for the phase shift data in Fig. 8.

density profile which has many peaks and valleys. In this case absolute densities of these peaks and valleys are chosen to be independent variables and normalization of densities of the contour lines between a peak and the neighboring valley is carried out with respect to the difference between the densities of the peak and valley. In a practical situation we do not know the number of extrema before calculation, but with physical consideration a relatively small number of calculations are sufficient to find out the best approximated profile.

As for the structure of the space of the unknown parameters, we cannot exclude the possibilities that many extrema of the objective function are found. As we have few generally reliable algorithm which determines the meaningful extremum among them, we have not tried to use an algorithm for a global problem. However, in the course of execution of the above examples we could usually reach the most appropriate extremum from the physical point of view. Though all the results in this paper are obtained using the nonlinear programming code of the simplex type, it is needless to say that some other codes, especially recently developed ones, may carry out the analysis more effectively from the viewpoint of computation time.

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REFERENCES

1. C. FLEURIER AND J. CHAPELLE, *Comput. Phys. Comm.* **7** (1974), 200, for example.
2. F. G. TRICOMI, "Integral Equations," p. 3, Interscience New York, 1957, for example.
3. E. P. GORBUNOV, YU. N. DNESTROVSKI, AND D. P. KOSTOMAROV, *Sov. Phys. Tech. Phys.* **13** (1968), 609.
4. J. KOWALIK AND M. R. OSBORNE, "Methods for Unconstrained Optimization Problems," American Elsevier, New York, 1968, for example.
5. K. TOI AND T. TAKEDA, *Nucl. Fusion* **14** (1974), 577.
6. K. TOI AND T. TAKEDA, Numerical method to obtain an optimum configuration of external magnetic field coils in tokamak device by non-linear programming, Japan Atomic Energy Research Institute Rep. JAERI-M6018, 1975.
7. J. A. NELDER AND R. MEAD, *Comput. J.* **7** (1965), 308.
8. S. KONOSHIMA AND N. FUJISAWA, private communication.